

# ENTROPY ESTIMATION AND FLUCTUATIONS OF HITTING AND RECURRENCE TIMES FOR GIBBSIAN SOURCES

(Running title: Entropy estimation and fluctuations of hitting times)

J.-R. CHAZOTTES<sup>a</sup>, E. UGALDE<sup>b</sup>

<sup>a</sup> CPHT-CNRS

École Polytechnique, 91128 Palaiseau Cedex, France

Email: [jeanrene@cpht.polytechnique.fr](mailto:jeanrene@cpht.polytechnique.fr)

<sup>b</sup> IICO - UASLP

A. Obregón 64, 78000 San Luis Potosí, SLP, México

Email: [ugalde@cactus.iico.uaslp.mx](mailto:ugalde@cactus.iico.uaslp.mx)

## Abstract

Motivated by entropy estimation from chaotic time series, we provide a comprehensive analysis of hitting times of cylinder sets in the setting of Gibbsian sources. We prove two strong approximation results from which we easily deduce pointwise convergence to entropy, lognormal fluctuations, precise large deviation estimates and an explicit formula for the hitting-time multifractal spectrum. It follows from our analysis that the hitting time of a  $n$ -cylinder fluctuates in the same way as the inverse measure of this  $n$ -cylinder at ‘small scales’, but in a different way at ‘large scales’. In particular, the Rényi entropy differs from the hitting-time spectrum, contradicting a naive ansatz. This phenomenon was recently numerically observed for return times that are more difficult to handle theoretically. The results we obtain for return times, though less complete, improve the available ones.

**Keywords and phrases:** hitting time, return time, non-overlapping return time, thermodynamic formalism, Gibbs measures, entropy estimation, Rényi entropy, multifractal spectrum, exponential law, central limit theorem, law of iterated logarithm, large deviations, intermittent map.

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# 1 Introduction

The setting of this work is a model for the following situation: One has time series  $x_1, x_2, x_3, \dots, x_N, \dots$  assumed to be typical realizations of some dynamical system. We further assume that the invariant measure of the system is Gibbsian. We then want to evaluate the entropy, the Rényi spectrum and the hitting-time/return-time spectrum of the system by using estimators based on hitting and return times. Our aim is to analyse the fluctuation properties of these estimators to have an *a priori* control on what we compute in practice. Regarding the estimator of the hitting-time/return-time spectrum, the issue is even to determine to what it converges.

Before being more specific and describing these estimators, let us temporally adopt a general point of view on hitting and return times. The first result in the early Ergodic Theory of dynamical systems is Poincaré's recurrence Theorem, see e.g. [21] or virtually any textbook on ergodic theory. Colloquially it states that in any dynamical system preserving a finite measure, typical orbits have a non-trivial recurrence behavior to each set of positive measure in that they come back infinitely often to it. More formally, let  $T : X \rightarrow X$  be a measurable transformation of the set  $X$ , the 'phase space', and  $\mu$  a  $T$ -invariant probability measure on  $X$ . Poincaré recurrence Theorem asserts that if  $A \subset X$  is a measurable set of positive measure, then there is a subsequence  $\{n_i\}_{i \geq 1}$  of the positive integers such that  $T^{n_i}x \in A$  for  $\mu$ -almost every point  $x \in A$ . One can also ask whether the orbit of a point  $x \notin A$  will eventually enter  $A$ . A sufficient condition is ergodicity: for  $\mu$ -almost every  $x$ , there is a finite time  $n_0 = n_0(x)$  such that  $T^{n_0}x \in A$ . This is the first hitting time.

It is natural to seek for quantitative descriptions of this recurrence and hitting time. Kac's Lemma, see e.g. [21], states that the conditional expectation (first moment) of the first recurrence time to  $A$  is equal to  $1/\mu(A)$ . This agrees with the intuition that the smaller the measure of  $A$  is, the longer it takes to come back. But ergodicity alone is not sufficient in general to guarantee the finiteness of the expectation of the first hitting time. Sufficient conditions on the mixing properties of the dynamics were recently given to ensure the finiteness of finitely many or all moments of hitting and return times [9]. We shall use them below.

To go further, we turn our attention to sets  $A$  which are cylinder sets. Let  $\mathcal{A}$  be a generating partition of  $X$ . To each point  $x \in X$  we associate its  $n$ -cylinder,  $\mathcal{A}_n(x)$ , defined as the intersection of all the elements of  $\mathcal{A}, T^{-1}\mathcal{A}, \dots, T^{-n+1}\mathcal{A}$  containing  $x$ . Let us phrase the following remarkable result [33]: The return time  $\mathbf{r}_n(x)$  of a  $\mu$ -typical point  $x$  into  $n$ -cylinder  $\mathcal{A}_n(x)$ , behaves as follows<sup>1</sup>:

$$\mathbf{r}_n(x) \asymp \exp(nh_\mu(T)) \quad (1)$$

where  $h_\mu(T)$  is the measure-theoretic entropy of the system. An analog result is available for the first time  $\mathbf{w}_n(x, y)$  the orbit of  $x$  enters the  $n$ -cylinder  $\mathcal{A}_n(y)$  about a point  $y$ , where  $x$  and  $y$  are 'randomly chosen' according to  $\mu$  independently of one another:

$$\mathbf{w}_n(x, y) \asymp \exp(nh_\mu(T)) \quad (2)$$

(But ergodicity is not enough: 'strong' mixing properties are necessary [29].) Formulas (1) and (2) give a simple entropy estimator based on the observation of a single typical orbit of the system.

It is natural to look for the asymptotic law of return times, that is to study, for each  $t > 0$ ,

$$\mu \left\{ z \in \mathcal{A}_n(x) : \mathbf{r}_n(z) \leq \frac{t}{\lambda_{\mathcal{A}_n(x)}} \right\} \quad (3)$$

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<sup>1</sup>The symbol ' $b_n \asymp c_n$ ' precisely means  $\lim_{n \rightarrow \infty} \frac{1}{n} \log b_n = \lim_{n \rightarrow \infty} \frac{1}{n} \log c_n$  ( $\mu$  or  $\mu \times \mu$  almost surely for pointwise quantities).

when  $n \rightarrow \infty$ . An easy heuristic argument shows that one must rescale  $t$  by a quantity proportional to  $\mu(\mathcal{A}_n(x))$  to obtain a non-trivial limiting law. Typically

$$\mu(\mathcal{A}_n(x)) \asymp \exp(-nh_\mu(T)) \quad (4)$$

by Shannon-McMillan-Breiman Theorem.

Under an ever lessening set of hypotheses on the nature and the speed of mixing of the system, the limiting law has been proved to be the exponential law both for hitting and return times, see for instance [4, 18].

The purpose of this paper is to analyse the fluctuations of return times and hitting times by using approximations by the inverse measure of cylinders. Indeed, combining (1) and (4), or (2) and (4) we get

$$\mathbf{r}_n(x) \asymp \frac{1}{\mu(\mathcal{A}_n(x))} \quad \text{or} \quad \mathbf{w}_n(x, y) \asymp \frac{1}{\mu(\mathcal{A}_n(x))} \quad (5)$$

Our aim is to sharpen these rough relations which are only pointwise. More precisely, we want to compare the fluctuations of these quantities. Do they have the same log-normal fluctuations? The same large deviations? On another hand, there is a fundamental class of ergodic measures, namely Bowen-Gibbs measures [7, 19], such that <sup>2</sup>

$$\mu(\mathcal{A}_n(x)) \sim \exp(S_n\varphi(x)) \quad (6)$$

where  $S_n\varphi(x)$  is the ‘energy’ of the cylinder  $\mathcal{A}_n(x)$ . Therefore, one can reduce the study of  $\log \mathbf{r}_n(x)$  or  $\log \mathbf{w}_n(x, y)$  to that of  $S_n\varphi(x)$ , which is much more easy to tackle and, indeed, all is known on the fluctuations of  $S_n\varphi(x)$ .

A related issue is to compute the so-called multifractal spectrum or Rényi entropy [25], defined as

$$\int \mu(\mathcal{A}_n(x))^{-q} d\mu(x) \asymp \exp(nq\overline{M}(q)) \quad \text{as } n \rightarrow \infty. \quad (7)$$

In view of practical computation of the Rényi entropy, when one only has at hand a time series, it is tempting to make the ansatz

$$\mu(\mathcal{A}_n(x))^{-1} \leftrightarrow \mathbf{r}_n(x)$$

in (7) and to evaluate the integral as a Birkhoff average. This was done by Grassberger [16]. The implicit assumption is that these two quantities have the same *large fluctuations*, as we shall explain below. An even more problematic point is the tacit assumption that *all moments* of the Poincaré recurrences are *finite* before taking the thermodynamic limit. The first explicit introduction of such a Poincaré recurrence spectrum is done in [17]. On the basis of numerical computations and heuristic arguments, it is claimed that the Poincaré recurrence spectrum and the Rényi spectrum do not coincide even in the setting of Bowen-Gibbs measures: They argue that the Poincaré recurrence spectrum must behave like  $1/q$  when  $q \rightarrow -\infty$ , which is not the case for the Rényi spectrum. Our goal is to clarify this claim in view of practical estimation of these spectra.

We will mainly concentrate on hitting times because they are simpler to analyse and, at the same time, do share the same properties with return times for strongly mixing measures like Bowen-Gibbs measures. Our tools are thermodynamical formalism and a very sharp result [1] that gives the error term in the convergence of (3) to the exponential law both in the size of  $\mathcal{A}_n$  and in  $t$ . From this we derive two

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<sup>2</sup>The symbol  $b_n \sim c_n$  means that  $\max(b_n/c_n, c_n/b_n)$  is bounded from above. We shall use it in the sequel.

approximation results: a global one and a local one. Indeed, Theorem 3.1 gives an approximation, for *any*  $n$ , of

$$\int \mathbf{w}_n^q d\mu \times \mu \quad \text{for all } q \in \mathbb{R} \quad (8)$$

as certain partition functions; Theorem 3.2 gives an almost-sure approximation of

$$\log(\mathbf{w}_n(x, y)\mu(\mathcal{A}_n(x))). \quad (9)$$

From (8) we deduce our large deviation results and an *explicit formula* for the hitting-time spectrum, whereas from (9) we derive a central limit theorem and even a law of iterated logarithm.

As a matter of fact, we shall see that  $(1/n)\log \mathbf{w}_n(x, y)$  has the same normal fluctuations as  $-(1/n)\log \mu(\mathcal{A}_n(x))$ , but *their large deviations are not the same* in some region. This is because

$$\begin{aligned} \int \mathbf{w}_n^q d\mu \times \mu &\sim \sum_{\mathcal{A}_n} \mu(\mathcal{A}_n)^{1-q}, \quad \text{for } q > -1 \\ \int \mathbf{w}_n^q d\mu \times \mu &\sim \sum_{\mathcal{A}_n} \mu(\mathcal{A}_n)^2, \quad \text{for } q \leq -1. \end{aligned}$$

This behavior is numerically observed in [17] for return times.

Concerning return times, we can completely analyse lognormal fluctuations but not get an explicit formula for the return-time spectrum. Namely we prove that this spectrum coincides with the Rényi spectrum on  $[0, +\infty)$ . The reason for this is the presence of ‘too soon recurrent cylinders’. Nevertheless, at the end of the paper we study non-overlapping return times  $\hat{\mathbf{r}}_n$ . For them we prove that  $\int \hat{\mathbf{r}}_n^q d\mu$  becomes flat for  $q < -1$  and coincides with the Rényi spectrum for  $q \in [0, \infty[$ . We conjecture that, in fact, the return-time spectrum really coincides with the hitting-time spectrum for Bowen-Gibbs measures.

**Outline of the paper.** In Section 2 we record relevant definitions and results on hitting times as well as on Bowen-Gibbs measures. In Section 3, we establish the two main theorems of the paper for hitting times, namely a strong *global* approximation of the ‘free energy’ of hitting times of  $n$ -cylinders, and a strong *local* approximation of hitting times. In Section 4, we derive a number of corollaries from these two theorems: pointwise convergence, precise large deviation estimates, an explicit formula for the hitting time spectrum, a central limit theorem and a law of iterated logarithm. In Section 5 we deal with return times. Section 6 contains three subsections. One is about (non-overlapping) return times. We improve our previous results by considering non-overlapping return times. Another one is concerned with bibliographical notes and possible straightforward extensions of our work. The last one illustrates that for the Manneville-Pomeau intermittent map, the hitting-time and return-time spectra are infinite for  $q \geq q_c$ , where  $q_c > 0$  (but we have a finite invariant measure).

## 2 Set-up and background

The phase space  $X$  will be the set of sequences  $x = (x_1, x_2, \dots)$  where  $x_j \in \mathcal{A}$  (the *finite* alphabet), that is  $X = \mathcal{A}^{\mathbb{N}}$ . The dynamics is given by the shift map  $T$  defined as  $(Tx)_j = x_{j+1}$  for all  $j \geq 1$ . We only consider full shifts since the passage to subshifts of finite type is not an issue. Given  $a_1^n \stackrel{\text{def}}{=} a_1 a_2 \dots a_n$ ,  $a_j \in \mathcal{A}$ , we denote by

$[a_1^n]$  the corresponding cylinder set that is  $[a_1^n] = \{x \in X : x_j = a_j, j = 1, 2, \dots, n\}$ . A point  $x \in X$  defines a sequence of cylinders that we naturally denote by  $[x_1^n]$ ,  $n \geq 1$ . The notation  $x_i^j$ ,  $1 \leq i \leq j$ , stands for  $x_i x_{i+1} \dots x_j$ . The natural  $\sigma$ -algebra  $\mathfrak{B}$  we take is the  $\sigma$ -algebra generated by cylinder sets. We omit to mention it in the sequel since it will always be the reference  $\sigma$ -algebra.

DEFINITION 2.1. We define the (first) hitting time of  $x$  to a cylinder  $[a_1^n]$  as follows:

$$\mathbf{t}_{[a_1^n]}(x) \stackrel{\text{def}}{=} \inf\{j \geq 1 : x_j^{j+n-1} = a_1^n\},$$

and the following hitting time, given  $x, y \in X$ :

$$\mathbf{w}_n(x, y) \stackrel{\text{def}}{=} \mathbf{t}_{[x_1^n]}(y) = \inf\{j \geq 1 : y_j^{j+n-1} = x_1^n\}$$

which is nothing but the (first) time one sees the  $n$  first symbols of  $x$  appearing in  $y$ , i.e., the first time that the orbit of  $y$  hits the cylinder  $[x_1^n]$ .

The time  $\mathbf{w}_n(x, y)$  is also called the waiting time [29].

Let us record the useful facts on the class of ergodic measures we are interested in. We refer the reader to [7, 24, 32] for full details. Let the potential  $\varphi : X \rightarrow \mathbb{R}$  be of ‘summable variations’. This means

$$\sum_{n \geq 1} \text{var}_n \varphi < \infty$$

where  $\text{var}_n \varphi \stackrel{\text{def}}{=} \sup\{|\varphi(x) - \varphi(y)| : x_1^n = y_1^n, x, y \in X\}$ . The condition imposed in [7] is more restrictive since it is  $\text{var}_n \varphi \leq C\theta^n$ , for some  $C > 0$ ,  $0 < \theta < 1$  (Hölder continuity).

BOWEN-GIBBS PROPERTY. Assume that  $\varphi : X \rightarrow \mathbb{R}$  has summable variations. Then there is a unique shift-invariant measure  $\mu = \mu_\varphi$ , that we call a Bowen-Gibbs measure, such that for all  $n \geq 1$ , for all  $a_1^n$  and for any  $x \in [a_1^n]$

$$K^{-1} \leq \frac{\mu([a_1^n])}{\exp(-nP(\varphi) + S_n \varphi(x))} \leq K \quad (10)$$

where  $K = K(\varphi) > 0$ ,  $S_n \varphi(x) \stackrel{\text{def}}{=} \varphi(x) + \varphi(Tx) + \dots + \varphi(T^{n-1}x)$  and  $P(\varphi)$  is the topological pressure of  $\varphi$ . From (10) it is easy to deduce that

$$P(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{a_1^n} \exp(S_n \varphi(a_1^n))$$

where  $S_n \varphi(a_1^n) \stackrel{\text{def}}{=} \sup\{S_n \varphi(x) : x_1^n = a_1^n\}$ . We assume without loss of generality that  $\varphi$  is normalized, which implies that  $P(\varphi) = 0$  and  $\varphi < 0$  [32]. A Bowen-Gibbs measure can also be characterized as an equilibrium state, that is the (unique) shift-invariant measure  $\eta$  that maximizes  $\int \varphi d\eta + h_\eta(T)$ , the maximum being equal to  $P(\varphi)$ . This is the Variational Principle. Since we assume  $P(\varphi) = 0$ , this leads to

$$-\int \varphi d\mu = h_\mu(T). \quad (11)$$

For any  $q \in \mathbb{R}$ , define

$$\mathfrak{M}(q) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{a_1^n} \mu([a_1^n])^{1-q} = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int \mu([x_1^n])^{-q} d\mu(x).$$

Using (10), we trivially have that

$$\mathcal{M}(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int e^{-q S_n \varphi(x)} d\mu(x). \quad (12)$$

It can be easily showed that  $q \mapsto \mathcal{M}(q)$  is a well-defined function on  $\mathbb{R}$  when  $\mu$  is a Bowen-Gibbs measure. Moreover this function is convex and increasing. Indeed, by using the Bowen-Gibbs property (10) and the definition of topological pressure, we easily get  $\mathcal{M}(q) = P((1-q)\varphi)$  (recall that  $P(\varphi) = 0$ ).

We now state the key result allowing us to analyse fluctuations of hitting times. The first result (with its proof) can be found in [1].

**KEY-LEMMA 2.1 (EXPONENTIAL DISTRIBUTION OF HITTING TIMES WITH ERROR TERM).** *Assume that  $\mu$  is a Bowen-Gibbs measure. Then there exist strictly positive constants  $c, C, \rho_1, \rho_2$ , with  $\rho_1 \leq \rho_2$ , such that for all  $n \in \mathbb{N}$ , all cylinder  $[a_1^n]$  and all  $t > 0$  there exists  $\rho(a_1^n) \in [\rho_1, \rho_2]$  such that one has*

$$\left| \mu \left\{ z : \mathbf{t}_{[a_1^n]}(z) > \frac{t}{\mu([a_1^n])} \right\} - \exp(-\rho(a_1^n)t) \right| \leq \varepsilon(a_1^n, t), \quad (13)$$

where  $\varepsilon(a_1^n, t) \stackrel{\text{def}}{=} C \exp(-cn) \exp(-\rho(a_1^n)t)$ .

**REMARK 1.** *The previous result is established under the hypothesis of  $\psi$ -mixing of the process. Bowen-Gibbs measures indeed have this property. When the potential is Hölder continuous, the proof of this property is done in [7] and in fact the  $\psi$ -mixing coefficient decreases exponentially fast. When the potential has summable but not exponentially small variations, the  $\psi$ -mixing property is established implicitly in the proof of Theorem 3.2 in [32]. Notice that in this case we do not know how fast the  $\psi$ -mixing coefficient decreases.*

### 3 Strong approximations of hitting times

We can state the main theorem of this section. For sequences of real numbers  $(b_n)$ ,  $(c_n)$ , the notation  $b_n \sim c_n$  precisely means that  $\max\{b_n/c_n, c_n/b_n\}$  is bounded from above.

We have the following

**THEOREM 3.1.** *Let  $\mu$  be a Bowen-Gibbs measure. Then*

$$\int \mathbf{w}_n^q d\mu \times \mu \sim \sum_{a_1^n} \mu([a_1^n])^{1-q}, \quad \text{for } q > -1 \quad (14)$$

and

$$\int \mathbf{w}_n^q d\mu \times \mu \sim \sum_{a_1^n} \mu([a_1^n])^2, \quad \text{for } q \leq -1. \quad (15)$$

*Proof.* Let  $q > 0$ . Then

$$\int \mathbf{w}_n^q d\mu \times \mu = \sum_{a_1^n} \mu([a_1^n]) \int \mathbf{t}_{[a_1^n]}^q d\mu \quad (16)$$

$$= q \sum_{a_1^n} \mu([a_1^n])^{1-q} \int_{\mu([a_1^n])}^{\infty} t^{q-1} \mu \left\{ \mathbf{t}_{[a_1^n]} > \frac{t}{\mu([a_1^n])} \right\} dt. \quad (17)$$

By Key-lemma 2.1 there exist positive constants  $A, B$  such that for any  $t > 0$  one has

$$\mu \left\{ \mathbf{t}_{[a_1^n]} > \frac{t}{\mu([a_1^n])} \right\} \leq A e^{-Bt}.$$

Key-lemma 2.1 also easily gives the lower bound :

$$\int_{\mu([a_1^n])}^{\infty} t^{q-1} \mu \left\{ \mathbf{t}_{[a_1^n]} > \frac{t}{\mu([a_1^n])} \right\} dt \geq K' - C \exp(-cn) K''$$

where  $0 < K' \stackrel{\text{def}}{=} \int_1^{\infty} t^{q-1} e^{-\rho_2 t} dt < \infty$  and  $0 < K'' \stackrel{\text{def}}{=} \int_0^{\infty} t^{q-1} e^{-\rho_1 t} dt < \infty$ . There exists an integer  $n_0$  such that for all  $n \geq n_0$ ,  $K' - C \exp(-cn) K'' > 0$ . Therefore we get

$$K_1 \sum_{[a_1^n]} \mu([a_1^n])^{1-q} \leq \int \mathbf{w}_n^q d\mu \times \mu \leq K_2 \sum_{a_1^n} \mu([a_1^n])^{1-q},$$

where

$$K_1 = q (K' - C \exp(-cn_0) K''), \quad K_2 = qA \int_0^{\infty} t^{q-1} e^{-Bt} dt.$$

This establishes (14) for  $q \geq 0$ . (The case  $q = 0$  is trivial.)

Let now  $q \in (-1, 0)$ .

$$\begin{aligned} \int \mathbf{w}_n^{-|q|} d\mu \times \mu &= \sum_{a_1^n} \mu([a_1^n]) \int \mathbf{t}_{[a_1^n]}^{-|q|} d\mu \\ &= \sum_{a_1^n} \mu([a_1^n]) \int_0^1 \mu \left\{ \mathbf{t}_{[a_1^n]}^{-|q|} > t \right\} dt \\ &= |q| \sum_{a_1^n} \mu([a_1^n])^{1+|q|} \int_{\mu([a_1^n])}^{\infty} t^{-|q|-1} \mu \left\{ \mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])} \right\} dt. \end{aligned} \quad (18)$$

We first obtain a lower bound for the integral in the last expression :

$$\int_{\mu([a_1^n])}^{\infty} t^{-|q|-1} \mu \left\{ \mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])} \right\} dt \geq \int_1^{\infty} t^{-|q|-1} \mu \left\{ \mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])} \right\} dt.$$

Hence, what matters is only the behavior for “large  $t$ ”. Using again Key-lemma 2.1, we get

$$\mu \left\{ \mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])} \right\} \geq 1 - (1 + C e^{-cn}) e^{-\rho_1 t}.$$

For all  $n \geq n_1$ , where  $n_1 \stackrel{\text{def}}{=} \frac{1}{c} \log \frac{e^{\rho_1}-1}{C} + 1$ , we have  $1 - (1 + C e^{-cn}) e^{-\rho_1} > 0$ . Since  $1 - (1 + C e^{-cn}) e^{-\rho_1 t} \geq 1 - (1 + C e^{-cn}) e^{-\rho_1}$  for all  $t \geq 1$ , we obtain, for all  $n \geq n_1$ ,

$$\int_1^{\infty} t^{-|q|-1} \mu \left\{ \mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])} \right\} dt \geq \frac{(1 - (1 + C e^{-cn_1}) e^{-\rho_1})}{|q|}.$$

We now turn to the upper bound. We obviously have

$$\int_{\mu([a_1^n])}^{\infty} t^{-|q|-1} \mu \left\{ \mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])} \right\} dt \leq \left( \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\infty} \right) t^{-|q|-1} \mu \left\{ \mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])} \right\} dt.$$

The integral from  $\frac{1}{2}$  to  $\infty$  is finite. Now we observe that, for every  $0 < t \leq \frac{1}{2}$ , we have the following estimate:

$$\mu \left\{ \mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])} \right\} \leq 1 - e^{-\rho_2 t}.$$

This estimate follows from the following lemma which is found in [1, Lemma 9].

LEMMA 3.1. For any integer  $t$  with  $t\mu([a_1^n]) \leq 1/2$ , one has

$$\rho_1 \leq -\frac{\log \mu\{\mathbf{t}_{[a_1^n]} > t\}}{t\mu([a_1^n])} \leq \rho_2,$$

where  $\rho_1, \rho_2$  are the (strictly positive) constants appearing in Key-lemma 2.1.

Using this estimate for the integral running from 0 to  $\frac{1}{2}$  we get a finite upper bound since

$$\int_0^{\frac{1}{2}} \frac{1 - e^{-\rho_2 t}}{t^{|q|+1}} dt < \infty.$$

Hence we conclude that

$$K'_1 \sum_{a_1^n} \mu([a_1^n])^{1+|q|} \leq \int \mathbf{w}_n^{-|q|} d\mu \times \mu \leq K'_2 \sum_{a_1^n} \mu([a_1^n])^{1+|q|},$$

where  $K'_1$  and  $K'_2$  are strictly positive constants. Hence we obtain (14) for  $q \in (-1, 0)$ .

Finally, let us consider the remaining case  $q \leq -1$ . Then for sufficiently large  $n$  (such that  $\mu([a_1^n]) < 1/2$ ), one has

$$\begin{aligned} \int \mathbf{w}_n^{-|q|} d\mu \times \mu &= |q| \sum_{a_1^n} \mu([a_1^n])^{1+|q|} \int_{\mu([a_1^n])}^{\infty} t^{-|q|-1} \mu\left\{\mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])}\right\} dt \\ &= |q| \sum_{a_1^n} \mu([a_1^n])^{1+|q|} \left[ \int_{\mu([a_1^n])}^{1/2} + \int_{1/2}^{\infty} \right] t^{-|q|-1} \mu\left\{\mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])}\right\} dt \\ &= |q| \sum_{a_1^n} \mu([a_1^n])^{1+|q|} [I_1(n, a_1^n) + I_2(n, a_1^n)]. \end{aligned}$$

Clearly the second integral  $I_2(n, a_1^n)$  is uniformly bounded in  $n$ . Indeed,

$$I_2(n, a_1^n) \leq \int_{1/2}^{\infty} \frac{1}{t^{1+|q|}} dt < +\infty.$$

However, the first integral  $I_1(n, a_1^n)$  is diverging when  $n \rightarrow \infty$ . Therefore the limiting behavior as  $n \rightarrow \infty$  is determined by

$$\begin{aligned} |q| \sum_{a_1^n} \mu([a_1^n])^{1+|q|} I_1(n, a_1^n) &= \\ |q| \sum_{a_1^n} \mu([a_1^n])^{1+|q|} \int_{\mu([a_1^n])}^{1/2} \mu\left\{\mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])}\right\} \frac{dt}{t^{1+|q|}}. \end{aligned}$$

We again use Lemma 3.1 to get

$$1 - e^{-\rho_1 t} \leq \mu\left\{\mathbf{t}_{[a_1^n]} \leq \frac{t}{\mu([a_1^n])}\right\} \leq 1 - e^{-\rho_2 t}.$$

provided that  $t \leq \frac{1}{2}$ . Using (10) (and  $P(\varphi) = 0$ ) we obtain

$$K^{-1} \exp(-c'n) \leq \mu([a_1^n]) \leq K \exp(-cn)$$

where  $c, c' > 0$ . Hence we get

$$I_1(n, a_1^n) \leq \rho_2 \int_{\mu([a_1^n])}^{\frac{1}{2}} t^{-|q|} dt \leq \frac{\rho_2(1 - 2^{|q|-1} K^{-1} e^{-c'n})}{|q| - 1} \mu([a_1^n])^{-|q|+1}$$



where we used the fact that for all  $\kappa \in \mathbb{R}$ ,  $1 - e^{-\kappa} \leq \kappa$ . Notice that for  $n$  large enough, the term between parentheses is strictly positive. Now, using the fact that  $1 - e^{-\kappa} \geq \kappa/2$  for any  $\kappa \in [0, 1]$ , and remembering that  $\rho_1/2 \leq 1$  <sup>(3)</sup>, and using again the Gibbs property (10), we obtain

$$I_1(n, a_1^n) \geq \frac{\rho_1(1 - 2^{|q|-1}Ke^{-cn})}{2(|q| - 1)} \mu([a_1^n])^{-|q|+1}$$

where the term between parentheses is strictly positive provided that  $n$  is sufficiently large. Therefore, for  $n$  large enough, we end up with

$$\frac{|q|\rho_1(1 - 2^{|q|-1}Ke^{-cn})}{2(|q| - 1)} \leq \frac{\int \mathbf{w}_n^{-|q|} d\mu \times \mu}{\sum_{a_1^n} \mu([a_1^n])^2} \leq \frac{2|q|\rho_2(1 - 2^{|q|-1}K^{-1}e^{-c'n})}{|q| - 1}.$$

(Notice that L'Hôpital's rule shows that there is no problem at  $q = -1$ .) Thus, we obtain (15), which finishes the proof.  $\square$

We now turn to local strong approximation estimates.

**THEOREM 3.2.** *Assume that  $\mu$  is a Bowen-Gibbs measure. Then there exists  $\epsilon_0 > 1$  such that for any  $\epsilon > \epsilon_0$ , one has*

$$-\epsilon \log n \leq \log(\mathbf{w}_n(x, y)\mu([x_1^n])) \leq \log \log(n^\epsilon) \quad (19)$$

eventually  $\mu \times \mu$ -a.s. <sup>(4)</sup>.

*Proof.* We want to find a summable upper-bound to

$$\begin{aligned} \mu \times \mu \{(x, y) : \log(\mathbf{w}_n(x, y)\mu([x_1^n])) > \log t\} = \\ \sum_{a_1^n} \mu([a_1^n]) \mu \{\log(\mathbf{t}_{[a_1^n]}\mu([a_1^n])) > \log t\} \end{aligned} \quad (20)$$

where  $t$  will be suitably chosen.

First observe that the function  $\epsilon(a_1^n, t) \leq Ce^{-cn}$  for all  $t > 0$  in (13). Throughout this proof this error bound will be sufficient for our purposes. Using (13) in (20) yields

$$\mu \times \mu \{(x, y) : \log(\mathbf{w}_n(x, y)\mu([x_1^n])) > \log t\} \leq Ce^{-cn} + e^{-\rho_1 t}.$$

Take  $t = t_n = \log(n^\epsilon)$ , where  $\epsilon > 0$  is to be chosen later on, to get

$$\mu \times \mu \{(x, y) : \log(\mathbf{w}_n(x, y)\mu([x_1^n])) > \log \log(n^\epsilon)\} \leq Ce^{-cn} + \frac{1}{n^{\rho_1 \epsilon}}.$$

Choose  $\epsilon > 1/\rho_1$ . An application of the classical Borel-Cantelli lemma tells us that

$$\log(\mathbf{w}_n(x, y)\mu([a_1^n])) \leq \log \log(n^\epsilon) \quad \text{eventually a.s.}$$

For the lower bound, observe that using (13) with the same simplified error bound as before, we get for all  $t > 0$

$$\mu \times \mu \{(x, y) : \log(\mathbf{w}_n(x, y)\mu([x_1^n])) \leq \log t\} \leq Ce^{-cn} + 1 - e^{-\rho_2 t} \leq Ce^{-cn} + \rho_2 t.$$

Choose  $t = t_n = n^{-\epsilon}$ ,  $\epsilon > 1$ , to get, proceeding as before,

$$\log(\mathbf{w}_n(x, y)\mu([x_1^n])) > -\epsilon \log n \quad \text{eventually a.s.}$$

The proof is finished by observing that both bounds hold for any  $\epsilon > \max(1, \rho_1)$ .  $\square$

<sup>3</sup>Indeed,  $\rho_2 = 2$ , see [1].

<sup>4</sup>By “eventually  $\mu \times \mu$ -a.s.” we mean that there exists a set  $G$  with  $\mu \times \mu(G) = 1$  and such that for any  $z \in G$  there is an integer  $\tilde{N} = \tilde{N}(z)$  such that for all  $n \geq \tilde{N}$  the inequality holds.

## 4 Corollaries

In this section we derive the corollaries of Theorems 3.1 and 3.2.

### 4.1 Almost-sure convergence

The following result tells us the ‘typical’ behavior of  $\mathbf{w}_n(x, y)$ . Throughout,  $h_\mu(T)$  is the (Kolmogorov-Sinai) entropy of  $(X, T, \mu)$ .

**COROLLARY 4.1 (ALMOST-SURE CONVERGENCE OF HITTING TIMES).** *Let  $\mu$  be a Bowen-Gibbs measure. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{w}_n(x, y) = h_\mu(T) \quad \text{for } \mu \times \mu - \text{a.e. } (x, y) .$$

*Proof.* By Theorem 3.2 we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{w}_n(x, y) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu([x_1^n]) = h_\mu(T) \quad \mu \times \mu - \text{a.e.}$$

where the second equality is given by Shannon-McMillan-Breiman Theorem.  $\square$

This result means that if we pick up randomly and independently of one another  $x$  and  $y$ , then the time needed for the orbit of  $y$  to hit  $[x_1^n]$  is typically of order  $\exp(nh_\mu(T))$ . In fact, this result is valid under the more general assumption that the process is weak Bernoulli (or  $\beta$ -mixing). For the details, we refer to [29]. Bowen-Gibbs measures are weak Bernoulli processes, see [7] for the proof.

**Remark.** In [8] the author assumes that  $x$  is picked up randomly according to an ergodic measure  $\eta$  whereas  $y$  is randomly (and independently) chosen according to a Bowen-Gibbs measure  $\mu$ . The previous result becomes:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{w}_n(x, y) = h_\eta(T) + h_T(\eta|\mu) \quad \text{for } \eta \times \mu - \text{a.e. } (x, y)$$

where  $h_T(\eta|\mu)$  is the relative entropy of  $\eta$  with respect to  $\mu$ . The results obtained in this paper could be suitably generalized to that situation.

### 4.2 Large deviations and multifractal spectra

In this section, we study large deviations of  $\frac{1}{n} \log \mathbf{w}_n$  around the entropy  $h_\mu(T)$  where  $\mu$  is a Bowen-Gibbs measure, that is, we only assume that  $\varphi$  has summable variations.

The ansatz consisting in replacing  $\mu([x_1^n])$  in the Rényi entropy by  $1/\mathbf{w}_n(x, y)$  leads to the following definition.

$$\mathcal{W}(q) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \log \int \mathbf{w}_n^q(x, y) d(\mu \times \mu)(x, y), \quad (21)$$

provided the limit exists. We do not use exactly the same definitions as in [17]. The present definitions are motivated by large deviation theory.

Introduce, for convenience, the following functions:

$$\mathcal{W}_n(q) \stackrel{\text{def}}{=} \frac{1}{n} \log \int \mathbf{w}_n^q(x, y) d(\mu \times \mu)(x, y)$$

for all  $n \geq 1$  and  $q \in \mathbb{R}$ . (Notice that  $\mathcal{W}_n(q)$  can be infinite.)

Observe that each function  $q \mapsto \mathcal{W}_n(q)$  is a convex (increasing) function on  $\mathbb{R}$  (hence, in particular, a continuous one <sup>5</sup>).

Observe also that

$$\int \mathbf{w}_n^q(x, y) d(\mu \times \mu)(x, y) = \sum_{a_1^n} \mu([a_1^n]) \int \mathbf{t}_{[a_1^n]}^q(y) d\mu(y)$$

There is no Kač formula for hitting times for general ergodic dynamical systems. Ergodicity only ensures that almost surely there is a finite first hitting-time. The only fact we know without any assumption is that  $\int \mathbf{t}_{[a_1^n]}^q d\mu < \infty$  for all  $q \leq 0$ . Indeed,

$$\int \mathbf{t}_{[a_1^n]}^q d\mu \leq 1$$

for any  $q \leq 0$ . From [9] it follows that for any Bowen-Gibbs measure and for any cylinder  $[a_1^n]$ , we have

$$\int \mathbf{t}_{[a_1^n]}^q d\mu < \infty \quad \text{for all } q \in \mathbb{R}.$$

Hence  $\mathcal{W}_n(q) < \infty$  for all  $q \in \mathbb{R}$ ,  $n \geq 1$ .

We now turn to large deviation results. We refer the reader to [11] for background on this topic.

**COROLLARY 4.2 (SCALED GENERATING FUNCTION OF HITTING TIMES).** *Assume that  $\mu$  is a Bowen-Gibbs measure. Then*

$$\mathcal{W}(q) = \begin{cases} \mathcal{M}(q), & \text{for } q \geq -1, \\ P(2\varphi), & \text{for } q < -1, \end{cases} \quad (22)$$

*If  $\mu$  is not the measure of maximal entropy, then the function  $q \mapsto \mathcal{W}(q)$  is strictly convex on  $(-1, \infty)$ .*

*Proof.* Clearly (14) and (15) imply (22). □

Notice that  $q \mapsto \mathcal{W}(q)$  is continuous (as it must be for a convex function on  $\mathbb{R}$ ) but not differentiable at  $q = -1$ . Indeed, it can be easily checked that the right derivative at  $-1$  of  $\mathcal{W}$  is not zero but equal to  $-\int \varphi d\mu_{2\varphi} > 0$ , where  $\mu_{2\varphi}$  is the Bowen-Gibbs measure for the potential  $2\varphi$ .

**COROLLARY 4.3 (LARGE DEVIATIONS OF  $\mathbf{w}_n$ ).** *Let  $\mu$  be a Bowen-Gibbs measure which is not the measure of maximal entropy. Then for all  $u \geq 0$  we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\mu \times \mu) \left\{ \frac{1}{n} \log \mathbf{w}_n > h_\mu(T) + u \right\} = \inf_{q > -1} \{ -(h_\mu(T) + u)q + \mathcal{W}(q) \}$$

*and for all  $u \in (0, u_0)$ ,  $u_0 \stackrel{\text{def}}{=} |\lim_{q \downarrow -1} \mathcal{W}'(q) - h_\mu(T)|$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(\mu \times \mu) \left\{ \frac{1}{n} \log \mathbf{w}_n < h_\mu(T) - u \right\} = \inf_{q > -1} \{ -(h_\mu(T) - u)q + \mathcal{W}(q) \}$$

Notice that  $u_0 > h_\mu(T)$ , that is, we capture the large fluctuations of  $\log \mathbf{w}_n/n$  above and below  $h_\mu(T)$  since  $\mathcal{W}'(0) = h_\mu(T)$  (see the appendix for the proof).

---

<sup>5</sup>Remind that a convex function defined on a finite interval of the real line can be discontinuous only at the endpoints of that interval.

*Proof.* By Theorem 3.1 and (12) we immediately get that for any  $q > -1$

$$\mathcal{W}(q) = \mathcal{M}(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \int \exp(-q S_n \varphi) d\mu.$$

It can be easily deduced from [31] that the function  $q \mapsto P(q\varphi)$  is  $C^1$  and strictly convex if (and only if)  $\mu$  is not the measure of maximal entropy. In the Hölder continuous case, it is real analytic and also strictly convex if (and only if)  $\mu$  is not the measure of maximal entropy [24].

We can apply a large deviation result due to Plachky and Steinebach [26]. (Recall that a strictly convex differentiable function has a strictly increasing derivative.)  $\square$

Let us remark that when the measure is the one of maximum entropy, there are no large fluctuations which is not surprising.

The Rényi spectrum is defined here as  $\overline{\mathcal{M}}(q) \stackrel{\text{def}}{=} \mathcal{M}(q)/q$  and the hitting-time spectrum  $\overline{\mathcal{W}}(q) \stackrel{\text{def}}{=} \mathcal{W}(q)/q$ . We get  $h_\mu(T)$  for the value of these spectra at  $q = 0$  (using L'Hôpital's rule).

**COROLLARY 4.4.** *For any Bowen-Gibbs measure we have the following:*

$$\overline{\mathcal{W}}(q) = \overline{\mathcal{M}}(q) \quad \text{for } q \geq -1 \quad \text{and} \quad \overline{\mathcal{W}}(q) = P(2\varphi)/q \quad \text{for } q < -1.$$

### 4.3 Log-normal fluctuations

The purpose of this section is to show that  $\mathbf{w}_n(x, y)$  and  $1/\mu([x_1^n])$  have the same lognormal fluctuations for Bowen-Gibbs measures associated to Hölder continuous potentials. Namely, we prove a central limit theorem and a law of iterated logarithm.

We refer the reader to [24] for full details on the central limit theorem for Hölder continuous observables with respect to Bowen-Gibbs measures with a Hölder continuous potential. Define the following variance:

$$\sigma^2 \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \int \left( S_n \varphi - n \int \varphi d\mu \right)^2 d\mu. \quad (23)$$

It is well-known that if  $\sigma^2 > 0$  ( $\sigma^2 < \infty$  because of the exponential decay of correlations) and

$$\forall t \in \mathbb{R} \quad \lim_{n \rightarrow \infty} \mu \left\{ \frac{-S_n \varphi - n h_\mu(T)}{\sigma \sqrt{n}} < t \right\} = \mathcal{N}(0, 1)((-\infty, t]) \quad (24)$$

where  $\mathcal{N}(0, 1)((-\infty, t]) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp(-\xi^2/2) d\xi$ . This is just a particular instance of the central limit theorem for Bowen-Gibbs measures [24] where the observable is  $-\varphi$ . By (11)  $\int -\varphi d\mu = h_\mu(T)$ . One has  $\sigma^2 = 0$  if and only if  $-\varphi + h_\mu(T)$  (or equivalently  $\varphi - \int \varphi d\mu_\varphi$ ) is a coboundary, i.e. a function of the form  $\varrho - \varrho \circ T$ , for some measurable function  $\varrho$ . This means that  $\sigma^2 = 0$  if and only if  $\mu$  is the (unique) measure of maximal entropy.

**COROLLARY 4.5.** *Assume that  $\mu$  is a Bowen-Gibbs measure with a Hölder continuous potential which is not the measure of maximal entropy. Then*

$$\lim_{n \rightarrow \infty} \mu \times \mu \left\{ \frac{\log \mathbf{w}_n - n h_\mu(T)}{\sigma \sqrt{n}} < t \right\} = \mathcal{N}(0, 1)((-\infty, t]). \quad (25)$$

Moreover,

$$\sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \int (\log \mathbf{w}_n - h_\mu(T))^2 d(\mu \times \mu). \quad (26)$$

*Proof.* Our goal is to show that the central limit theorem for  $\log \mathbf{w}_n(x, y)$  results from the one for  $-\log \mu([x_1^n])$  which in turn results from the one for  $-S_n \varphi(x)$ . The latter assertion is trivial since, for all  $x$ ,

$$-S_n \varphi(x) - C \leq -\log \mu([x_1^n]) \leq -S_n \varphi(x) + C \quad (27)$$

where  $C > 0$ , due to Bowen-Gibbs inequality (10). Hence the quantities  $-\log \mu([x_1^n])$  and  $-S_n \varphi(x)$  have the same mean  $h_\mu(T)$  and variance  $\sigma^2$ .

We now use the strong approximation formula (19) from Theorem 3.2 together with inequalities (27) to get

$$\frac{\log \mathbf{w}_n(x, y) + S_n \varphi(x)}{\sigma \sqrt{n}} \rightarrow 0 \quad \text{for } \mu \times \mu - \text{almost every } (x, y). \quad (28)$$

By a basic result of Probability Theory (see [13] for instance), the preceding result and (24) imply the desired statement.

The proof of (26) is given in the Appendix.  $\square$

We emphasize that (26) does not follow from Theorem 3.2, see the Appendix.

We now state and prove a law of iterated logarithm for  $\log \mathbf{w}_n$ :

**COROLLARY 4.6.** *Assume that  $\mu$  is a Bowen-Gibbs measure with a Hölder continuous potential which is not the measure of maximal entropy. Then*

$$\limsup_{n \rightarrow \infty} \frac{\log \mathbf{w}_n - nh_\mu(T)}{\sigma \sqrt{2n \log \log n}} = 1 \quad \mu \times \mu - \text{a.e.} \quad (29)$$

Remark that we get  $-1$  instead of  $1$  when taking ‘lim inf’ instead of ‘lim sup’. In fact, we could show that the set of accumulation points of the sequence  $\{(\log \mathbf{w}_n - nh_\mu(T))/\sqrt{2n \log \log n}\}_n$  is the interval  $[-\sigma, +\sigma]$ .

*Proof.* Using Theorem 3.2 we get that eventually  $\mu \times \mu$ -almost surely

$$-\frac{\epsilon \log n}{\sigma \sqrt{2n \log \log n}} \leq \frac{\log \mathbf{w}_n - nh_\mu(T)}{\sigma \sqrt{2n \log \log n}} - \frac{-S_n \varphi - nh_\mu(T)}{\sigma \sqrt{2n \log \log n}} \leq \frac{\log \log n^\epsilon}{\sigma \sqrt{2n \log \log n}}$$

Taking the limit supremum  $n \rightarrow \infty$  and using the law of iterated logarithm for  $-S_n \varphi$ , we finish the proof.  $\square$

The law of iterated logarithm for  $-S_n \varphi$  can be found in [12].

## 5 Return times

We now turn to return times. As we shall see, we obtain less complete results than for hitting times.

### 5.1 Set-up

**DEFINITION 5.1.** *The (first) return time of a point  $x$  into its  $n$ -cylinder  $[x_1^n]$ ,  $n \geq 1$  is defined as:*

$$\mathbf{r}_n(x) \stackrel{\text{def}}{=} \inf\{k \geq 2 : x_k^{k+n-1} = x_1^n\}.$$

The following result is proved in [2, Section 6]. In order to state it we need to define the set of  $n$ -cylinders with ‘internal periodicity’  $p \leq n$ :

$$\mathcal{S}_p(n) \stackrel{\text{def}}{=} \{[a_1^n] : \min\{k \in \{1, \dots, n\} : [a_1^n] \cap T^{-k}[a_1^n] \neq \emptyset\} = p\}.$$

Notice that the set of  $n$ -cylinders can be written as the union  $\bigcup_{1 \leq p \leq n} \mathcal{S}_p(n)$ .

KEY-LEMMA 5.1 (EXPONENTIAL DISTRIBUTION OF BASIC RETURN TIMES). *Let  $\mu$  be a Bowen-Gibbs measure. Then there exist strictly positive constants  $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$  such that for any  $n \in \mathbb{N}$ , any  $p \in \{1, \dots, n\}$ , any cylinder  $[a_1^n] \in \mathcal{S}_p(n)$ , one has for all  $t \geq p$*

$$\left| \mu \left\{ z : \mathbf{r}_n(z) > \frac{t}{\zeta(a_1^n) \mu([a_1^n])} \mid [a_1^n] \right\} - \zeta(a_1^n) \exp(-t) \right| \leq \tilde{\varepsilon}(a_1^n, t), \quad (30)$$

where  $\tilde{\varepsilon}(a_1^n, t) \stackrel{\text{def}}{=} \tilde{C}_1 \exp(-\tilde{C}_2 n) \exp(-\tilde{C}_3 t)$  and  $\zeta(a_1^n)$  is such that  $|\zeta(a_1^n) - \rho(a_1^n)| \leq D \exp(-\tilde{C}_2 n)$ , with  $D > 0$  and  $\rho(a_1^n)$  given in Key-lemma 2.1. Moreover,

$$\mu\{z : \mathbf{r}_n(z) > t \mid [a_1^n]\} = 1 \quad \text{for all } t < p.$$

## 5.2 Large deviations

We define the following (possibly infinite) quantities, for all  $q \in \mathbb{R}$ , provided the limit exists:

$$\mathcal{R}(q) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{1}{n} \log \int \mathbf{r}_n^q(z) d\mu(z) \quad (31)$$

$$\mathcal{R}_n(q) \stackrel{\text{def}}{=} \frac{1}{n} \log \int \mathbf{r}_n^q(z) d\mu(z) \text{ and}$$

Without any assumption on  $\mu$ , the function  $\mathcal{R}$  trivially exists at  $q = 0$  and equal zero. If  $\mu$  is assumed to be ergodic then  $\mathcal{R}(1) = \log |\mathcal{A}|$ . Indeed,

$$\int \mathbf{r}_n(z) d\mu(z) = \sum_{a_1^n} \left( \int \mathbf{r}_n(z) d\mu_{[a_1^n]}(z) \right) \mu([a_1^n]).$$

By Kač's formula the integral is equal to  $1/\mu([a_1^n])$ , hence we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \int \mathbf{r}_n(z) d\mu(z) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{a_1^n} 1 = \log |\mathcal{A}|.$$

Let us emphasize that the finiteness of  $\mathcal{R}$  for all  $q > 1$  is not obvious at all. We need to know the finiteness of  $\int_{[a_1^n]} \mathbf{r}_n^q d\mu$  for all  $q > 1$ , that is the finiteness of the moments of return times to  $[a_1^n]$ . This point seems to have been overlooked before. For  $q \leq 1$ , all moments of return times are of course finite due to Kač's formula ( $T$ -invariance is indeed sufficient) but nothing rules out *a priori* the possibility that the moment of the return time be infinite beyond a certain order  $q > 1$  for some  $n_0$  (and hence for all  $n \geq n_0$  since the moment of order  $q > 0$  as a function of  $n$  is increasing). This will be illustrated in Section 6.1. From [9] it follows that for any Bowen-Gibbs measure and for any cylinder  $[a_1^n]$ , we have

$$\int_{[a_1^n]} \mathbf{r}_n^q d\mu_{[a_1^n]} < \infty \quad \text{for all } q \in \mathbb{R}.$$

Hence  $\mathcal{R}_n(q) < \infty$  for all  $q \in \mathbb{R}, n \in \mathbb{N}$ .

PROPOSITION 5.1 (PARTIAL LARGE DEVIATIONS FOR  $\mathbf{r}_n$ ). *Let  $\mu$  be a Bowen-Gibbs measure which is not the measure of maximal entropy. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu \left\{ \frac{1}{n} \log \mathbf{r}_n > h_\mu(T) + u \right\} = \inf_{q \geq 0} \{ -(h_\mu(T) + u)q + \mathcal{R}(q) \}$$

that holds for all  $u \geq 0$ .

*Proof.* From Key-Lemma 5.1, we can deduce that

$$\int \mathbf{r}_n^q d\mu \sim \sum_{a_1^n} \mu([a_1^n])^{1-q}, \quad (32)$$

for all  $q > 0$  (the case  $q = 0$  is trivial). This implies that  $\mathcal{M}(q) = \mathcal{R}(q)$  for all  $q \geq 0$ . Thus we can again apply Plachky-Steinebach's large deviation estimate on  $\mathbb{R}^+$ . We leave the details of the proof of (32) to the reader since it is very similar to the one for hitting times we gave above.  $\square$

Some large deviation results are given in [10] for  $\mathbf{r}_n$ , but they are valid only in a small (non-explicit) interval around  $h_\mu(T)$ . This restriction is due to 'too soon recurrent cylinders'. Because of the same problem, we can only extend this result to the whole range of possible deviations above  $h_\mu(T)$ .

### 5.3 Lognormal fluctuations

We summarize what happens for return times in the following theorem and corollary.

**THEOREM 5.1.** *Assume that  $\mu$  is a Bowen-Gibbs measure. Then there exists  $\epsilon_1 > 1$  such that for any  $\epsilon > \epsilon_1$ , one has*

$$-\epsilon \log n \leq \log(\mathbf{r}_n(x) \mu([x_1^n])) \leq \log \log(n^\epsilon) \quad (33)$$

eventually  $\mu$ -a.s.

**COROLLARY 5.1.** *Let  $\mu$  be Bowen-Gibbs measure associated to Hölder continuous potential. Then Corollaries 4.5-4.6 hold for  $\mathbf{r}_n$  instead of  $\mathbf{w}_n$ . ( $\mu \times \mu$  is replaced by  $\mu$ .)*

**Sketch of proofs.** The proof of the corollary follows exactly the same lines as that for hitting times. Let us sketch the upper bound, leaving the lower bound to the reader. To apply the classical Borel-Cantelli lemma, as before, we need to upper bound

$$\mu \{x : \log(\mathbf{r}_n(x) \mu([x_1^n])) \geq \log t \mid [a_1^n]\} =$$

$$\sum_{p=1}^n \sum_{[a_1^n] \in \mathcal{S}_p(n)} \mu([a_1^n]) \mu \{ \log(\mathbf{r}_n \mu([a_1^n])) \geq \log t \mid [a_1^n] \} \quad (34)$$

where  $t$  will be suitably chosen. We now use Key-Lemma 5.1 to get

$$(34) \leq \sum_{p=1}^n \sum_{[a_1^n] \in \mathcal{S}_p(n)} \mu([a_1^n]) \left( (\rho_2 + D) e^{-\frac{\rho_1}{2} t} + \tilde{C}_1 e^{-\tilde{C}_2 n} \right)$$

for all  $t \geq p \mu([a_1^n])$ , where we used the fact that there is some  $n_0$  such that for all  $n \geq n_0$ ,  $\rho_1/2 \leq \rho_1 - D e^{-\tilde{C}_2 n} \leq \zeta(a_1^n) \leq \rho_2 + D$ . Now choose  $t = t_n = \log(n^\epsilon)$ ,  $\epsilon > 0$  and notice that  $t_n \geq p \mu([a_1^n])$ , for all  $p = 1, \dots, n$  for  $n$  large enough since  $\mu([a_1^n]) \leq K e^{-cn}$  by the Bowen-Gibbs property. Therefore we get

$$(34) \leq \underbrace{\sum_{p=1}^n \sum_{[a_1^n] \in \mathcal{S}_p(n)} \mu([a_1^n])}_{=1} \left( \frac{\rho_2 + D}{n^{\epsilon \rho_1/2}} + \tilde{C}_1 e^{-\tilde{C}_2 n} \right)$$

which is summable in  $n$  provided that  $\epsilon > 2/\rho_1$ . We leave the rest of the proof to the reader.

## 6 Final comments and open questions

### 6.1 Hitting-time and return-time spectra in presence of intermittency

We consider here the following intermittent map (the so-called Manneville-Pomeau map) defined on the interval  $[0, 1]$ :

$$T : x \mapsto x + x^{1+\alpha} \mod 1, \quad \alpha \in (0, 1).$$

The thermodynamic formalism for such a map is now well-understood. We refer the reader to the recent paper [22] for more details and references on what we will use.

Let  $\varphi = -\log|T'|$ , the potential function. The map  $T$  admits an absolutely continuous invariant measure  $\mu$  which is an equilibrium state for the potential  $\varphi$ . ( $\mu$  is not the only equilibrium state. Any measure of the form  $t\mu + (1-t)\delta_0$ , where  $t \in [0, 1]$  is an equilibrium state for  $\varphi$ ;  $\delta_0$  is the Dirac measure at 0.) This means that  $P(\varphi) = h_\mu(T) + \int \varphi d\mu$ , where  $P(\varphi)$  is the topological pressure of  $\varphi$  and  $h_\mu(T)$  is the measure-theoretic entropy. In fact  $P(\varphi) = 0$  because of the Rokhlin formula. It is well-known that  $\mathcal{M}(q) = P((1-q)\varphi)$ . From the behavior of the pressure function we get the following properties for  $\mathcal{M}(q)$ : it is continuous, convex and non-decreasing. Moreover,  $P((1-q)\varphi) = 0$  for  $q \leq 0$ ,  $P((1-q)\varphi) > 0$  for  $q > 0$  and  $q \mapsto P((1-q)\varphi)$  is a real-analytic function for  $q > 0$ . At the critical point one has the following asymptotics:

$$\frac{P((1-q)\varphi)}{q} \rightarrow h_\mu(T) \text{ as } q \downarrow 0.$$

The Manneville-Pomeau map has two intervals of monotonicity,  $I_0, I_1$ , from which one can define cylinder sets:  $I_{i_1, i_2, \dots, i_n}(x) = I_{i_1} \cap T^{-1}I_{i_2} \cap \dots \cap T^{-n+1}I_{i_n}$  is that interval of monotonicity for  $T^n$  which contains  $x$ . Contrarily to the case when maps are everywhere expanding, the ratio

$$\frac{\mu(I_{i_1, i_2, \dots, i_n}(x))}{\exp\left(\sum_{k=0}^{n-1} \varphi(T^k x)\right)} \quad (35)$$

is not uniformly bounded in  $n$  and  $x$ . This comes from the fact that distortions are not bounded. A more careful analysis shows that one can find bounds from above and below which are polynomial in  $n$  and uniform in  $x$ . Such a measure is an example of a weak Gibbs measure.

The following basic proposition shows that large deviation results in the usual sense do not hold.

**PROPOSITION 6.1.** *For all  $q \geq \frac{1}{\alpha}$ , we have for all  $n \geq 1$  that  $\mathcal{W}_n(q) = \infty$  (hence  $\mathcal{W}(q) = \infty$ , but  $\mathcal{M}(q)$  is finite for every  $q \in \mathbb{R}$ ). The same occurs for  $\mathcal{R}_n(q)$  for all  $q \geq \frac{1}{\alpha} + 1$ .*

*Proof.* We are going to show that  $\mathcal{W}_1(q) = \log \int \mathbf{w}_1^q d(\mu \times \mu)$  becomes infinite from some  $q_0 = q_0(\alpha) > 0$  on (hence  $\mathcal{W}_n(q) = \infty$  for all  $n$  when  $q \geq q_0$  since  $\mathbf{w}_{n+1}(x, y) \geq \mathbf{w}_n(x, y)$  for all  $(x, y)$ .) The main point is the following estimate for  $\mu(I_{0\dots 0})$ :

$$\mu(\underbrace{I_{0\dots 0}}_{\ell \text{ symbols}}) \geq C \ell^{-\frac{1}{\alpha}}$$

where  $C$  is a positive constant. But  $\mu(I_{0\dots 0})$  (with  $\ell$  symbols) is nothing but the measure of points that do not enter the right interval  $I_1$  before  $\ell$  iterations. Now



observe that

$$\int \mathbf{w}_1^q d\mu \times \mu \geq \mu(I_1) \int \mathbf{t}_{I_1}^q d\mu = \mu(I_1) \sum_{\ell \geq 0} [(\ell+1)^q - \ell^q] \mu(I_{\underbrace{0 \dots 0}_{\ell \text{ symbols}}}).$$

Therefore  $\mathcal{W}_n(q) = \infty$  for all  $n \geq 1$  and  $q \geq \frac{1}{\alpha}$ .

To pass from  $\mathcal{W}_n(q)$  to  $\mathcal{R}_n(q)$  use Proposition 1 in [9].  $\square$

It was recently proved in [15] that, for  $0 < \alpha < \frac{1}{2}$ ,  $1/\mu(I_{i_1, i_2, \dots, i_n}(x))$  and  $\mathbf{r}_n(x)$  have the same lognormal fluctuations

Proposition 6.1 shows that the return and hitting time spectra are not relevant for non-uniformly hyperbolic systems. Indeed, a single indifferent fixed point makes these spectra infinite for all  $q \geq q_c(\alpha)$ . Moreover, it implies that  $1/\mu(I_{i_1, i_2, \dots, i_n}(x))$  and  $\mathbf{r}_n(x)$  cannot have the same large deviations.

## 6.2 More on return times

In [17] the authors study the recurrence-time spectrum  $\mathcal{R}(q)$ . They show heuristically that  $\mathcal{R}(q)$  must behave like  $1/q$  as  $q \rightarrow \infty$ . (Be careful of the different convention used therein to define  $\mathcal{R}(q)$ .) Two numerical simulations confirm this behavior: The graphs of the spectrum really look like a constant divided by  $q$  for  $q \gtrsim 1$ . This is indeed what we get rigorously (remember that  $q = 2$  corresponds to  $q = -1$  in our convention) for hitting-time spectrum  $\mathcal{W}(q)$ . We are not able, as we saw above, to prove this for return times. The difficulty comes from ‘too soon recurrent’ cylinders. More technically speaking, we do not have the analog of Lemma 3.1 for return times and therefore the finiteness of the integral in (18) escapes us.

We can go a bit further by looking at non-overlapping return times  $\hat{\mathbf{r}}_n$  (studied recently in [5]). By definition, such return times cannot be ‘too small’: The (first) non-overlapping return time of a point  $x$  into its  $n$ -cylinder  $[x_1^n]$ ,  $n \geq 1$  is defined as:

$$\hat{\mathbf{r}}_n(x) \stackrel{\text{def}}{=} \inf\{k \geq 1 : x_{kn+1}^{(k+1)n} = x_1^n\}.$$

We have the following approximation result:

**PROPOSITION 6.2.** *Let  $\mu$  be a Bowen-Gibbs measure. Then, for every  $q < -1$ ,*

$$\int \hat{\mathbf{r}}_n^q d\mu \asymp \sum_{a_1^n} \mu([a_1^n])^2.$$

For every  $q \geq 0$ ,

$$\int \hat{\mathbf{r}}_n^q d\mu \asymp \sum_{a_1^n} \mu([a_1^n])^{1-q}.$$

(The symbol  $\asymp$  is precisely defined at the beginning of Section 3.) Therefore, if we let  $\hat{\mathcal{R}}(q)$  be the analog of  $\mathcal{R}(q)$  where  $\hat{\mathbf{r}}_n$  replaces  $\mathbf{r}_n$ , the previous result implies that

$$\hat{\mathcal{R}}(q) = P(2\varphi) \quad \text{for all } q < -1 \quad \text{and} \quad \hat{\mathcal{R}}(q) = \mathcal{M}(q) \quad \text{for all } q \geq 0$$

(Use the Bowen-Gibbs property and the definition of topological pressure of Section 2.)

Recall that for hitting times we proved a more precise result (Theorem 3.1) since the second approximation works for  $q \in [-1, \infty[$  in that case.

*Proof.* First write

$$\int \hat{\mathbf{r}}_n^q d\mu = \sum_{k=1}^{\infty} k^{-|q|} \mu\{\hat{\mathbf{r}}_n = k\}. \quad (36)$$

Using the  $\psi$ -mixing property (see [5] for details) we get that

$$\mu([a_1^n])(1 - \psi(k-1)) \leq \mu_{[a_1^n]}\{\hat{\mathbf{r}}_n = k\} \leq \mu([a_1^n])(1 + \psi(k-1)).$$

Hence,

$$\frac{1}{2} \sum_{a_1^n} \mu([a_1^n])^2 \leq \mu\{\hat{\mathbf{r}}_n = k\} \leq \frac{3}{2} \sum_{a_1^n} \mu([a_1^n])^2.$$

Therefore we get

$$\frac{1}{2} C_q \leq \frac{\int \hat{\mathbf{r}}_n^q d\mu}{\sum_{a_1^n} \mu([a_1^n])^2} \leq \frac{3}{2} C_q$$

where  $C_q \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^{-|q|} < \infty$  since  $|q| > 1$ . We leave the proof of the statement in the range  $q \in [0, \infty[$  to the reader (use the analog of Key-lemma 5.1 for  $\hat{\mathbf{r}}_n$  that can be extracted from [5]).

So, we arrived at the desired result.  $\square$

We conjecture that Theorem 3.1 is true with return-times instead of hitting-times. It could be easier to first prove this conjecture for non-overlapping return times.

### 6.3 Relevance of the hitting-time and return-time spectra

In view of [14] (saturation of level sets) and what we obtained in the present paper, one can legitimately ask what is the relevance of the return time spectrum, except as a trick to compute the Rényi spectrum for  $q > -1$ . The same can be said for the hitting-time spectrum. Even in the comfortable setting of Bowen-Gibbs measures, these spectra contain no information for  $q \leq -1$ . In presence of intermittency, we saw that they are infinite for  $q \geq q_c$ .

### 6.4 Related works and an extension

It is worth to indicate to the reader the differences between our work and the previous ones. In the paper [10], the authors study only return times for Bowen-Gibbs measures associated to a Hölder continuous potential. They prove a central limit theorem and a large deviation principle. Here we not only improve the lognormal approximation but also extend the range of accessible large deviations above the true entropy. Moreover we handle potentials with summable variations and not only Hölder continuous ones for large deviations. We also mention [30] for a general (but much less sharp) result on lognormal fluctuations for return times. In the context of  $\psi$ -mixing stochastic processes, there are two references [20, 34]. Both deal with local strong approximations, in particular central limit theorems. The author of [20] directly uses the  $\psi$ -mixing property. The author of [34] first proves an approximation to the exponential law of rescaled hitting and return times and then deduces strong local approximations. We emphasize that his approximation is much less sharp than the one we use here. This difference is not relevant for deriving strong local approximations but becomes essential to handle large deviations. The only paper dealing with large deviations of hitting times is [3] where the authors study the first occurrence of a cylindrical pattern in the realisation of a Gibbsian random field on the lattice  $\mathbb{Z}^d$ ,  $d \geq 2$ . Our proof is very similar to that of this work.

We also note that our results can be extended to a more general class of processes, namely the processes satisfying the  $\varphi$ -mixing property with a summable  $\varphi$ -mixing sequence. This is because Key-lemma 2.1 is proved not only for  $\psi$ -mixing but also for such processes [1]. But up to our knowledge, this does not define a natural class of equilibrium states on shift spaces. That is why we did not state our results under this assumption. Nevertheless, an interesting class of non-Markov expanding maps of the interval has this property with an exponentially decreasing  $\varphi$ -mixing sequence (with respect to the partition given by the discontinuity points of the map). This class was indeed studied in [23]. We could therefore sharpen the results of that paper since Corollary 5.1, apply. We could of course write down the analogous results for hitting times. Regarding large deviations of hitting times, we could derive some approximations in the spirit of Theorem 3.1 and derive some estimates like that of Corollary 4.3. But one has to be careful with the control of some ‘bad’ cylinders for which the “distorsion property” (the analog of Bowen-Gibbs property (10) in that context) does not hold, which is the price to be paid for the non-Markovian partition.

## 7 Appendix

We prove the convergence in mean and in quadratic mean of  $\log \mathbf{w}_n$ . The former is related to the slope of  $\mathcal{W}$  at 0, see the comment after Corollary 4.3. The latter is related to the proof of (26) in Corollary 4.5. We emphasize that Theorem 3.2 does not help because almost sure convergence does not say anything for  $L^p$  convergence unless we have bounded random variables, which is definitively not the case here.

It is easy to get that  $\mathcal{W}'_n(0) = \frac{1}{n} \int \log \mathbf{w}_n \, d\mu \times \mu$ . Since  $q \mapsto \mathcal{W}_n(q)$  is a convex and, at least, continuously differentiable function on  $(-1, +\infty)$ ,  $\mathcal{W}'(0) = \lim_{n \rightarrow \infty} \mathcal{W}'_n(0)$ . Let us show that  $\lim_{n \rightarrow \infty} \mathcal{W}'_n(0) = h_\mu(T)$ .

First observe that convergence in mean of  $(\log \mathbf{w}_n)_n$  is equivalent to showing

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int |\log(\mathbf{w}_n(x, y) \mu([x_1^n])| \, d\mu \times \mu(x, y) = 0.$$

Indeed,

$$\begin{aligned} & \frac{1}{n} \int |\log \mathbf{w}_n(x, y) - nh_\mu(T)| \, d\mu \times \mu(x, y) \leq \\ & \frac{1}{n} \int |\log(\mathbf{w}_n(x, y) \mu([x_1^n])| \, d\mu \times \mu(x, y) + \frac{1}{n} \int |\log \mu([x_1^n]) + nh_\mu(T)| \, d\mu(x). \end{aligned}$$

The second term goes to zero by using (10),  $P(\varphi) = 0$  and (11) and noting that  $\int S_n \varphi \, d\mu = n \int \varphi \, d\mu$  by  $T$ -invariance of  $\mu$ . We have

$$\int |\log(\mathbf{w}_n(x, y) \mu([x_1^n])| \, d\mu \times \mu(x, y) = \sum_{a_1^n} \mu([a_1^n]) \int_0^\infty \mu\{\log(\mathbf{t}_{[a_1^n]} \mu([a_1^n])) > t'\} \, dt'$$

The change of variable  $\log t = t'$  leads to

$$\sum_{a_1^n} \mu([a_1^n]) \int_1^\infty \mu\{\mathbf{t}_{[a_1^n]} > t/\mu([a_1^n])\} \frac{dt}{t}.$$

By Key-Lemma 2.1, this integral is finite and bounded between, say,  $C_1$  and  $C_2$  (independent of  $n$ ). Therefore we get the desired result.

Now turn to the convergence in quadratic mean of  $(\log \mathbf{w}_n/n)_n$ . Observe that the following identity holds:

$$\frac{1}{n} \int (\log \mathbf{w}_n - nh_\mu(T))^2 \, d(\mu \times \mu) =$$

$$\begin{aligned} & \frac{1}{n} \int (\log(\mathbf{w}_n \mu([x_1^n]))^2 d(\mu \times \mu) + \frac{1}{n} \int (\log \mu([x_1^n] + nh_\mu(T))^2 d\mu + \\ & - \frac{2}{n} \int \log(\mathbf{w}_n \mu([x_1^n])) (\log \mu([x_1^n] + nh_\mu(T)) d(\mu \times \mu) \end{aligned}$$

The second term goes to  $\sigma^2$  (see formula (23) and use (10)). Hence the proof is done if we show that the two other terms go to 0 as  $n \rightarrow \infty$ . Proceeding as before we get that the integral in the first term equals

$$2 \sum_{a_1^n} \mu([a_1^n]) \int_1^\infty \mu\{\mathbf{t}_{[a_1^n]} > t/\mu([a_1^n])\} \frac{\log t}{t} dt.$$

Using again Key-lemma 2.1, we bound the integral from above and below uniformly in  $n$ . Now consider the integral in the third term which is equal to

$$\sum_{a_1^n} \mu([a_1^n]) (\log \mu([a_1^n]) + nh_\mu(T)) \int \log(\mathbf{t}_{[a_1^n]}(y) \mu([a_1^n])) d\mu(y).$$

We recognize the same integral as above which we know bounded from above and below uniformly in  $n$ . The factor in front of the integral was also treated above in this section. The proof is finished.

We could prove the same results for  $\mathbf{r}_n$  by using Key-lemma 5.1 and for  $\hat{\mathbf{r}}_n$  by using the corresponding result found in [5].

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